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Complex Geometry, Unification, and Quantum Gravity. II. The Generations Problem

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The new model of elementary particles as vertical vectors on the principal fiber bundle $U(3, 2) \to U(3, 2)/U(3, 1) \times U(1)$ introduced in Part I is extended to higher generations.

1. INTERACTION VIA LIE BRACKET

The new theory of elementary particles introduced in Part I (Love, 1993) was shown to model successfully the interactions of the fundamental particles. In this theory, we model the particles as vertical vectors on the bundle $U(3, 2)/U(3, 1) \times U(1)$ and thus they can be locally represented as the product ft, where t is an element of the Lie algebra $U(3, 1) \times U(1)$ and f is an eigenfunction of the generalized Casimir operators of $U(3, 2)$. We call the vector t the algebraic factor and we call f the function factor of the particle. These vertical vectors are merely a geometric interpretation of the standard "operator valued distribution." In Part I, the algebraic factor of the fundamental particles was given. Let Z_{IJ} be the matrix with a 1 in the IJ position and zero elsewhere.

The communication rules are

$$[Z_{IJ}, Z_{KL}] = \delta_{JK} Z_{IL} - \delta_{IL} Z_{KJ}$$

The Z_{II} form the Cartan subalgebra and are spectrum generators:

$$[Z_{II}, Z_{IL}] = Z_{IL}, \qquad [Z_{II}, Z_{KI}] = -Z_{KI}$$

where the eigenvalues are the roots.

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The algebraic factors and the roots of the fundamental particles are then as given in Table I.

As shown in Part I, the roots of the Lie algebra will act as the internal quantum numbers. The Z_{11} eigenvalue is the lepton number, the Z_{22} eigenvalue is a new quantum number related to the spin, the Z_{33} eigenvalue is the baryon number, and the Z_{44} eigenvalue is the electric charge.

We now begin the task of identifying the algebraic factors of all elementary particles. In this paper, we extend the analysis of algebraic factors to include the "stable particles" of the Particle Data Group (1984).

If we can achieve this goal, it will be an indication that the model deserves further development. We will not attempt to identify the function factors of any particles, but leave that for future installments of this series of papers.

If A and B are particles with the same algebraic factor, we will write $A =_F B$ (read "A equals B modulo functions"). We will analyze the stable particle table of the particle data group one particle at a time. Once the algebraic factors of a few particles are known, decay modes yielding the known particles will permit the identification of the Lie algebra factor of the other particles and thus a bootstrapping through the tables of particle decays.

In analyzing multiparticle decay processes, it is awkward to compute brackets. Instead, we will add the roots of the known particles to obtain the roots of unknown particles. When computing brackets, the order of the factors is important; change the order and the product either changes sign or is zero. There are no such problems with adding the roots. This will allow a smooth transition to the more exotic interactions we will analyze later.

Table I

	Z_{11}	Z_{22}	Z_{33}	Z_{44}
$\nu = Z_{12}$	1	−1	0	0
$H = Z_{13}$	1	0	−1	0
$e^- = Z_{14}$	1	0	0	−1
$\bar{\nu} = Z_{21}$	−1	1	0	0
$n = Z_{23}$	0	1	−1	0
$\pi^- = Z_{24}$	0	1	0	−1
$\bar{H} = Z_{31}$	−1	0	1	0
$\bar{n} = Z_{32}$	0	−1	1	0
$p^- = Z_{34}$	0	0	1	−1
$e^+ = Z_{41}$	−1	0	0	1
$\pi^+ = Z_{42}$	0	−1	0	1
$p^+ = Z_{43}$	0	0	−1	1

To illustrate this point, let us compute the bracket $[e^-, \nu_e]$ using the roots

$$\begin{array}{lrrrr} e^- =_F Z_{14} & 1 & 0 & 0 & -1 \\ \bar{\nu}_e =_F Z_{21} & -1 & 1 & 0 & 0 \end{array}$$

Adding the roots, we obtain

$$0 \quad 1 \quad 0 \quad -1$$

which are the roots of Z_{24} as calculated directly from the matrix representation above. Thus, adding the roots identifies the product of the interaction, but does not yield the sign. As a further illustration of the method, let us analyze the oldest known decay, beta decay:

$$n \to p^+ e^- \bar{\nu}_e \tag{1.1}$$

The analysis of the decay (1.1) via roots reads

$$\begin{array}{lrrrr} e^- =_F Z_{14} & 1 & 0 & 0 & -1 \\ \bar{\nu}_e =_F Z_{21} & -1 & 1 & 0 & 0 \\ p^+ =_F Z_{43} & 0 & 0 & -1 & 1 \end{array}$$

Thus the sum of the roots of the right-hand side is

$$n =_F Z_{23} \quad 0 \quad 1 \quad -1 \quad 0$$

which are the roots of Z_{23}, the algebraic factor of left-hand side of (1.1). Thus, we confirm that the algebraic factor the neutron is Z_{23}.

Summing the roots works when there are no diagonal elements involved. For example, to compute $[e^-, e^+]$:

$$\begin{array}{lrrrr} e^- =_F Z_{14} & 1 & 0 & 0 & -1 \\ e^+ =_F Z_{41} & -1 & 0 & 0 & 1 \end{array}$$

Adding,

$$0 \quad 0 \quad 0 \quad 0$$

The roots being all zero, we know that the product is diagonal, but we do not know which diagonal element we have. In the same way, the interaction of any particle with its antiparticle yields diagonal elements:

$$[Z_{IJ}, Z_{JI}] = Z_{II} - Z_{JJ}$$

Supposedly, when a particle meets its antiparticle, the result is photons. In the algebra above, we see that the result depends on which type of particle–antiparticle pair interacts. These diagonal elements are the neutral currents (Okun, 1985, p. 53). The massless particles appear on the diagonal, and all off-diagonal particles except the neutrino are known to be massive. For consistency, we must expect the neutrinos to be massive, although we cannot predict their masses now.

The roots we are working with are the eigenvalues of Z_{II} ($I=1, 2, 3, 4$) in the adjoint representation. We would work with any other basis of the Cartan subalgebra of $u(3, 1)$. Ramakrishan (1972, 1980) used the linear combinations $I_{KL}=(Z_{KK}-Z_{LL})/2$ in his derivation of the generalized Gell–Mann–Nishijima relation. Ramakrishnan showed that the spectra of the I_{KL} are the proper generalizations of the isotopic spin numbers of $su(3)$. Thus, the fundamental numbers are the eigenvalues of the diagonal operators Z_{II}, but the linear combinations may also be important experimentally, as we saw with the case of spin in Part I.

2. EXCITED STATES

Continuing with the analysis of other particle decays, we will first work with excited states of the fundamental particles. To explain the similarities between the muon and the electron, these particles must have the same algebraic factor and differ only in the function factor. Since the Lie algebra factor accounts for all the particle interactions except gravity, the data confirm this observation: "Today we know that the muon behaves like a heavy electron and the hypothesis of muon-electron universality introduces the same interaction with the coupling constants for both muon and electron" (Morita, 1973, p. 237); "the muon is known from its magnetic moment to be correctly described as a 'heavy electron'" (Jauch and Rohrlich, 1976, pp. 536–537). For consistency, the two neutrons ν_e and ν_μ also must have the same algebraic factor. Then in the decay

$$\mu^- \rightarrow e^- \nu_e \bar{\nu}_\mu \tag{2.1}$$

the function factor of the μ^- decays into a lower energy function, the function factor of the e^-, while the excess energy goes to create what is essentially a particle–antiparticle pair. Since it is not exactly a particle–antiparticle pair, the correctness of this description of the decay is questionable, so further analysis is required. But this analysis can only be accomplished once we have the function factors in hand.

If the neutrinos have a mass, we would expect the heavier neutrinos (tau and mu) to decay into the electron neutrino. However, we should not expect neutrino oscillations in that the electron–neutrino will not turn spontaneously into a mu-neutrino.

Barut (1972) argues that the muon cannot be an excited electron since we do not observe the decay $\mu \to e\gamma$. According to the present picture, in (2.1) the neutrino–antineutrino pair is essentially a photon. This decay may occur with the γ in turn forming a neutrino–antineutrino pair, but in too short a time span for present-day technology to detect it.

The next decay up for analysis is

$$\pi^- \to \mu^- \bar{\nu}_\mu \tag{2.2}$$

The algebraic factors of all of the particles have been identified:

$$\mu^- =_F e^- =_F Z_{14} \quad 1 \quad 0 \quad 0 \quad -1$$
$$\bar{\nu}_\mu =_F \bar{\nu}_e =_F Z_{21} \quad -1 \quad 1 \quad 0 \quad 0$$

Adding,

$$\pi^- =_F Z_{24} \quad 0 \quad 1 \quad 0 \quad -1$$

Thus, the identification of the algebraic factor of the pion is confirmed. Continuing with the analysis of new decays, let us examine

$$K^+ \to \mu^+ \nu \tag{2.3}$$

Modulo functions, we just saw that $\mu^+ \nu_\mu =_F \pi^+$, so (2.3) implies that

$$K^+ =_F \pi^+ =_F Z_{42} \tag{2.4}$$

We analyze other decay routes of the K^+ in the Appendix.

The next decay leads to some interesting consequences and provides our first internal check for consistency:

$$\Lambda \to p^+ \pi^- \tag{2.5}$$

$$p^+ =_F Z_{43} \quad 0 \quad 0 \quad -1 \quad 1$$
$$\pi^- =_F Z_{24} \quad 0 \quad 1 \quad 0 \quad -1$$

Adding

$$\Lambda =_F Z_{23} \quad 0 \quad 1 \quad -1 \quad 0$$

Thus, Λ has the same Lie algebra factor as n. So from the decay

$$\Lambda \to n\pi^\circ \tag{2.6}$$

we conclude that π° has all-zero spectrum, i.e., the Lie algebra factor of the π° is diagonal. Since γ is known to be diagonal, this observation is consistent with the known decay

$$\pi^\circ \rightarrow 2\gamma \tag{2.7}$$

Since the strangeness number of the Λ is not zero, an interesting conclusion is that strangeness, like the muon number, is not an internal quantum number. This does not rule out the possibility that the strangeness could be an eigenvalue of one of the generalized Casimir operators of $U(3, 2)$. We limit the present classification of the particles to an analysis of the spectra of the four basis elements of the Cartan subalgebra, the roots of the Lie algebra. This is the easy part of the classification because the eigenvalues are those of matrices. To complete the classification will require that we know the generalized Casimir operators of $U(3, 2)$ and their eigenfunctions. Since the group is of rank 4, including the diagonal dilation operator, there are five such operators to deal with, or order 1, 2, 3, 4, and 5, respectively. Consequently, the task is highly nontrivial. We expect these Casimir operators to yield invariants corresponding to mass, momenta, magnetic moment, plus new quantum numbers. In this paper, only the first steps are taken as a justification for pursuing the entire program.

We now turn to the task of finding the Lie algebra factor of the particles in the Stable Particle Table of the Particle Data Group's Tables of Particle Properties. We tabulate this information in the Appendix with an analysis of other particle decay schemes. Let us now analyze

$$\Sigma^+ \rightarrow n\pi^+ \tag{2.8}$$

$$\begin{array}{lrrrr} n =_F Z_{23} & 0 & 1 & -1 & 0 \\ \pi^+ =_F Z_{42} & 0 & -1 & 0 & 1 \end{array}$$

Adding

$$\Sigma^+ =_F Z_{43} \quad 0 \quad 0 \quad -1 \quad 1$$

which is the same algebraic factor as the proton. Given this observation, decays of the Σ^+ into a proton plus diagonal decay products—a set of particles whose net product is diagonal (the sum of the roots is zero)—should occur. This may be a particle whose algebraic factor is diagonal or a particle–antiparticle pair, or something more complicated. Since the π° is diagonal, such a decay is

$$\Sigma^+ \rightarrow p^+\pi^\circ \tag{2.9}$$

Since γ is diagonal, the observed decay

$$\Sigma^+ \rightarrow \Lambda\gamma \tag{2.10}$$

implies $\Sigma^\circ =_F \Lambda =_F n =_F Z_{23}$.

To find the Lie algebra factor of the particles with many decay routes, I will select the decay that is the easiest to analyze. The results tabulated in the Appendix show the consistency of this analysis.

The three decay modes of the τ,

$$\begin{aligned} &\tau^- \rightarrow \pi^- \nu \\ &\tau^- \rightarrow \rho^- \nu \\ &\tau^- \rightarrow K^- \nu \end{aligned} \tag{2.11}$$

allow us to conclude that the π^-, ρ^-, and K^- all have the same algebraic factor:

$$\pi^- =_F \rho^- =_F K^- =_F Z_{24} \quad 0 \quad 1 \quad 0 \quad -1$$

For the daughter particles in (2.11), the roots are

$$\begin{array}{lrrrr} \pi^- =_F Z_{24} & 0 & 1 & 0 & -1 \\ \nu_e =_F Z_{12} & 1 & -1 & 0 & 0 \end{array}$$

Adding, we obtain the roots of the parent:

$$\tau^- =_F Z_{14} \quad 1 \quad 0 \quad 0 \quad -1$$

The decays

$$\eta \rightarrow e^+ e^- \tag{2.12}$$

$$K^\circ \rightarrow e^+ e^- \tag{2.13}$$

involve a particle–antiparticle pair and calculation of the bracket reveals that

$$\eta =_F K^\circ =_F Z_{11} - Z_{44}$$

Since K° is diagonal, the decay

$$D^+ \rightarrow K^\circ \pi^+ \tag{2.14}$$

allows us to conclude that

$$D^+ =_F \pi^+ =_F Z_{42}$$

Since D° decays into a particle–antiparticle pair,

$$D^\circ \rightarrow \pi^+ \pi^- \tag{2.15}$$

we calculate the bracket to obtain

$$D^\circ =_F Z_{22} - Z_{44}$$

Because η is diagonal, the decay

$$F^+ \to \eta p^+ \tag{2.16}$$

implies that

$$F^+ =_F \pi^+ =_F Z_{42}$$

Since D° is diagonal, from the decay

$$B^+ \to D^\circ \pi^+ \tag{2.17}$$

we conclude

$$B^+ =_F \pi^+ =_F Z_{42}$$

The particle–antiparticle pair appearing in the decay

$$B^\circ \to D^\circ \pi^+ \pi^- \tag{2.18}$$

implies that $B^\circ =_F D^\circ =_F Z_{22} - Z_{44}$.

A diagonal daughter assists us in one more analysis:

$$\Lambda_c^+ \to p^+ K^\circ \tag{2.19}$$

Since K° is diagonal, we must have

$$\Lambda_c^+ =_F p^+$$

Since π° is diagonal, the decay

$$\Xi^\circ \to \pi^\circ \Lambda$$

implies

$$\Xi =_F \Lambda =_F Z_{23}$$

3. SOME COMPLICATIONS

All of the particles analyzed thus far were excited states of one of the fundamental particles. The particles analyzed from here on are not of this simple form; instead they are composite particles, some being excited states of nuclei.

Since γ is diagonal, the decays

$$\Xi^- \to \Sigma^- \gamma \tag{3.1}$$

$$\Omega^- \to \Xi^- \gamma \tag{3.2}$$

imply

$$\Xi^- =_F \Sigma^-, \qquad \Omega^- =_F \Xi^- \tag{3.3}$$

which shows that the Σ^-, Ξ^-, and Ω^- all have the same algebraic factor. Identification of this factor would be easy if Σ^- were the antiparticle of Σ^+, but it is not. The analysis to this point has been effortless; at this stage we encounter our first subtlety. Analysis of the Σ^- decay routes will reveal the mystery:

$$\Sigma^- \to n\pi^- \tag{3.4}$$

The algebraic factors and roots of the right-hand side of (3.4) are

$$\begin{array}{lrrrr} n =_F Z_{23} & 0 & 1 & -1 & 0 \\ \pi^- =_F Z_{24} & 0 & 1 & 0 & -1 \end{array}$$

Adding,

$$0 \quad 2 \quad -1 \quad -1$$

which is not an entry in Table I.

$$\Sigma^- \to ne^-\bar{\nu}_e \tag{3.5}$$

The algebraic factors and roots of the right-hand side of (3.5) are

$$\begin{array}{lrrrr} n =_F Z_{23} & 0 & 1 & -1 & 0 \\ e^- =_F Z_{14} & 1 & 0 & 0 & -1 \\ \bar{\nu}_e =_F Z_{21} & -1 & 1 & 0 & 0 \end{array}$$

Adding,

$$0 \quad 2 \quad -1 \quad -1$$

which again is the same as (3.4), but is not an entry in Table I. The Ξ^- decays yield the same algebraic quantum numbers:

$$\Xi^- \to \Lambda\pi^- \tag{3.6}$$

The algebraic factors and roots of the right-hand side of (3.6) are

$$\begin{array}{lrrrr} \Lambda =_F Z_{23} & 0 & 1 & -1 & 0 \\ \pi^- =_F Z_{24} & 0 & 1 & 0 & -1 \end{array}$$

Adding,

$$0 \quad 2 \quad -1 \quad -1$$

which is again the same set of numbers, but not an entry in Table I.

$$\Xi^- \rightarrow \lambda e^- \bar{\nu}_e \tag{3.7}$$

The algebraic factors and roots of the right-hand side of (3.7) are

$$\begin{array}{rrrrr} \Lambda =_F Z_{23} & 0 & 1 & -1 & 0 \\ e^- =_F Z_{14} & 1 & 0 & 0 & -1 \\ \bar{\nu} =_F Z_{21} & -1 & 1 & 0 & 0 \end{array}$$

Adding,

$$0 \quad 2 \quad -1 \quad -1$$

which is the same set of numbers, but not an entry in Table I.

$$\Xi^- \rightarrow p^+ \pi^- \pi^- \tag{3.8}$$

The algebraic factors and roots of the right-hand side of (3.8) are

$$\begin{array}{rrrrr} p^+ =_F Z_{43} & 0 & 0 & -1 & 1 \\ \pi^- =_F Z_{24} & 0 & 1 & 0 & -1 \\ \pi^- =_F Z_{24} & 0 & 1 & 0 & -1 \end{array}$$

Adding,

$$0 \quad 2 \quad -1 \quad -1$$

The numbers obtained from these various decays are consistent, but not an entry in Table I. Evidently we have the correct numbers for three particles. The decays of the Ω^- serve to reinforce this conclusion:

$$\Omega^- \rightarrow \Lambda K^- \tag{3.9}$$

The algebraic factors and roots of the right-hand side are

$$\begin{array}{rrrrr} \Lambda =_F Z_{23} & 0 & 1 & -1 & 0 \\ K^- =_F Z_{24} & 0 & 1 & 0 & -1 \end{array}$$

Adding,

$$0 \quad 2 \quad -1 \quad -1$$

Next, $\Omega^- \rightarrow \Xi^\circ \pi^-$

$$\begin{array}{rrrrr} \Xi^\circ =_F Z_{23} & 0 & 1 & -1 & 0 \\ \pi^- =_F Z_{24} & 0 & 1 & 0 & -1 \end{array} \tag{3.10}$$

Adding,

$$0 \quad 2 \quad -1 \quad -1$$

So we have confirmed these numbers as the algebraic quantum numbers of these three particles. But they are not entries in Table I. Every set of numbers encountered heretofore was in that table. The numbers in Table I are those numbers arising when two particles interact via the Lie bracket. Obviously some other model for the interaction is necessary at this stage. Earlier we saw another type of interaction which preserves the quantum numbers: the tensor product. The numbers for the Σ^- are consistent with

$$\begin{aligned} \Sigma^- &= n \otimes \pi^- \\ &=_F Z_{23} \otimes Z_{24} \end{aligned} \tag{3.11}$$

These numbers are again consistent with the decay

$$K^- p^+ \to \Omega^- K^+ K^\circ \tag{3.12}$$

The roots of the left-hand side are

$$\begin{array}{lrrrr} K^- =_F Z_{24} & 0 & 1 & 0 & -1 \\ p^+ =_F Z_{43} & 0 & 0 & -1 & 1 \\ & 0 & 1 & -1 & 0 \end{array}$$

while the algebraic quantum numbers of the right-hand side are

$$\begin{array}{lrrrr} \Omega^- =_F Z_{23} \otimes Z_{24} & 0 & 2 & -1 & -1 \\ K^+ =_F Z_{42} & 0 & -1 & 0 & 1 \\ & 0 & 1 & -1 & 0 \end{array}$$

showing that the numbers agree before and after the interaction. This analysis raises another question: Which other particles require the tensor product for their description?

The decay

$$\Lambda_c^+ \to K^- \Delta^{++} \tag{3.13}$$

with

$$\begin{array}{lrrrr} \Lambda_c^+ =_F Z_{43} & 0 & 0 & -1 & 1 \\ K^- =_F Z_{24} & 0 & 1 & 0 & -1 \end{array}$$

implies that Δ^{++} has the algebraic quantum numbers

$$0 \quad -1 \quad -1 \quad 2$$

Again, this is not an entry in Table I. The double charge also would indicate that Δ^{++} is different from anything so far encountered. Again, these quantum numbers can be obtained from a tensor product:

$$\Delta^{++} =_F Z_{42} \otimes Z_{23} \otimes Z_{42} \tag{3.14}$$

We previously analyzed

$$\Omega^- =_F Z_{23} \otimes Z_{24}$$

The same quantum numbers are obtained with

$$\Omega^- =_F Z_{24} \otimes Z_{43} \otimes Z_{24} \tag{3.15}$$

With (3.14) and (3.15) as evidence, the following conjecture is obvious: particles interact via the "tensor force" via an "exchange" of one factor. Thus, a particle of type $A =_F B \otimes C \otimes D$ is possible iff C can interact via bracket with both B and D. Then we have three ways to obtain the same algebraic quantum numbrs:

$$A =_F B \otimes C \otimes D$$

$$A =_F B \otimes [C, D]$$

$$A =_F [B, C] \otimes D$$

This leads to the further conjecture that nuclei bond together by the tensor interaction with protons interchanging particles with the same algebraic factor as the pion such as the W. Since $n =_F [p^+, \pi^-]$, the nucleus of the deuteron is

$$n \otimes p =_F [p^+, \pi^-] \otimes p^+ =_F p^+ \otimes [\pi^-, p^+] =_F p^+ \otimes \pi^- \otimes p^+$$

We see that protons in the nucleus react by exchanging *real* (not virtual) pions. Thus, the new model of matter has tremendous implications for nuclear physics. We see that some of the "particles" now thought to be elementary are composite. Thus, there is no clear line between nuclear and particle physics. In the late 1930s there were several papers along this line. Many physicists visualized the nucleus as protons exchanging various particles. This model fell into disfavor with most physicists who, believing in particle democracy, felt that the neutron was as fundamental as the proton and hence the nucleus consisted of protons and neutrons held together by the exchange of some other (virtual) particles. I am thus advocating the return to the older viewpoint. Kursunoglu (1979) and Barut (1972, 1981) have also advocated this return and the interested reader should refer to their papers for the history and further consequences of these ideas.

The above description of interchange deals with the case where a change of particle type occurs. There are many interactions in nature where there is no change in particle type. What does the present model have to say about these interactions? Fortunately, they fit into the "tensor" force pattern where the exchanged particles are diagonal (i.e., neutral currents). Thus, to describe the electromagnetic interaction of two protons we could write

$$[p^+, \gamma_4]\otimes p^+ =_F p^+\otimes[\gamma_4, p^+]$$

or, in terms of the Lie algebra factors

$$[Z_{43}, Z_{44}]\otimes Z_{43} =_F Z_{43}\otimes[Z_{44}, Z_{43}]$$

The electromagnetic interaction of a proton with an electron may be described by

$$[Z_{14}, Z_{44}]\otimes Z_{43} =_F Z_{14}\otimes[Z_{44}, Z_{43}]$$

Thus the exchange force, familiar from QED, remains in the present model. With a suitable geometric reinterpretation, much of standard quantum field theory will carry over to the present setting. Exactly what does and what does not carry over will be the subject of future papers in this series.

4. CONCLUSION

The analysis of this paper shows that the scheme introduced in Part I is at least consistent and promising. There are many questions remaining. The use of vertical vector fields instead of Lie algebra-valued forms has provided some interesting insights. As we have shown, it is the eigenvalue associated with the Lie algebra factor of the vertical vector field that determines whether or not that particle interacts via a given force. For gravitation to fit into this pattern, the mass, as the gravitational quantum number, must be derived from the spectrum of the remaining diagonal operator, Z_{55}, which would then be the "graviton." This operator commutes with all the elements of $u(3, 1)$, i.e., with all of the algebraic factors of the particles, and thus would interact only with the function factor. Every vertical vector field has a function factor and thus all particles will interact via gravitation.

The present model has one thing in common with the quark model; both are based on group theory. The original three-quark model was based on SU(3), with the quarks being a basis for the three-dimensional space on which $SU(3)$ acts. Although $SU(3)$ is a subgroup of $U(3, 1)$, there are no quarks in this model, because here the elements of $U(3, 1)$ act on themselves rather than some "internal space." The quark model includes only the hadrons; the present model includes the leptons as well and comes closer to a truly unified picture. The quark model classifies particles into triplets, octets, etc., primarily on the basis of their mass. The present model classifies the particles on the basis of their interactions. This led to four superselection rules which in turn led to 16 fundamental families of particles. Because we have not identified the differential equations satisfied by the function factors, the present model does not yet have the predictive power of the quark model. But some avenues of research are clearly laid out.

The physics derived from the geometry must reflect the fact that the eigenvalue determines the strength of the interaction. Thus, scaling the geometry will be important. We will accomplish this goal by assigning units to the structure constants of $u(3, 2)$.

Specifying the Lie algebra factor the particles provides a classification based on half of the information that will ultimately be available. Classification based on the Lie algebra factor has accounted for the four superselection rules: spin, baryon number, lepton number, and electric charge (Emerson, 1972). Further classification based on the generalized Casimir operators of $U(3, 2)$ is required. We expect that analysis to lead to further relations between the function factor and the algebraic factor of the particle. Once that is done, we will be able to calculate the masses and the transition probabilities. This problem was treated in a similar setting by Barut and Kleinert (1967*a*–*c*) and by Herrick and Sinanoglu (1972). In the models based on compact groups, the Wigner–Eckhart theorem shows that analysis of the transition probabilities requires the Clebsch–Gordan coefficients for the group. But the Wigner–Eckhart theorem is not valid for noncompact groups, and consequently the analysis is neither routine nor straightforward.

From work done by other researchers, it is clear that the second-order Casimir operator will account for the mass–energy relationship. The role of the higher-order Casimir operators is not clear simply because there is no precedent theory with differential operators of order three, four, and five.

APPENDIX. THE STABLE PARTICLES

Using the results of the text, we compare the data from the Tables of Particle Properties Stable Particle Table with the model. In the following table of decay modes, the daughter particles enclosed in double brackets [[*]] are of the same algebraic type as the present, and those enclosed in double parentheses ((*)) form a diagonal decay product. Recall that specifying the Lie algebra factor of the particle is classifying the particles based on only half the ultimate information. The Lie algebra factors lead to superselection rules. The algebraic quantum numbers account for the four superselection rules listed by Emerson (1972): spin, electric charge, baryon number, and lepton number.

ν_e	
ν_μ	
ν_τ	
τ^-	$[[\mu^-]]((\nu\nu))$
	$[[e^-]]((\nu\nu))$
	$[[\pi^-\nu]]$
	$[[\rho^-\nu]]$
	$[[K^-\nu]]$
	$[[K^{*-}(892)\nu]]$
	$[[K^{*-}(1430)\nu]]$
	$[[\pi^-\nu]](\rho^\circ))$
	$[[\mu^-]]((\gamma))$
	$[[e^-]]((\gamma))$
	$[[\mu^-]]((\mu^-\mu^+))$
	$[[e^-]]((\mu^-\mu^+))$
	$[[\mu^-]]((e^-e^+))$
	$[[e^-]]((e^-e^+))$
	$[[\mu^-]]((\pi^\circ))$
	$[[e^-]]((\pi^\circ))$
	$[[\mu^-]]((K^\circ))$
	$[[e^-]]((K^\circ))$
	$[[\mu^-]]((\rho^\circ))$
	$[[e^-]]((\rho^\circ))$
π^+	$[[\mu^+\nu]]$
	$[[e^+\nu]]$
	$[[\mu^+\nu]]((\gamma))$
	$[[e^+\nu]]((\gamma))$
	$[[e^+\nu]]((\pi^\circ))$
	$[[e^+\nu]]((e^+e^-))$
	$[[\mu^+\nu]]$
π°	$[[\gamma]]((\gamma))$
	$[[\gamma]]((e^+e^-))$
	$[[\gamma]]((\gamma))((\gamma))$
	$[[e^+e^-]]((e^+e^-))$
	$[[\gamma]]((\gamma))((\gamma))((\gamma))$
	$[[e^+e^-]]$

$[[\nu\bar{\nu}]]$
$[[e^+e^-]]((\mu^-\mu^+))$
η° $[[\gamma]]((\gamma))$
$[[\pi^\circ]]((\pi^\circ))((\pi^\circ))$
$[[\pi^\circ]]((\gamma))((\gamma))$
$[[\pi^\circ]]((\pi^+\pi^-))$
$((\pi^+\pi^-))[[\gamma]]$
$[[\gamma]]((e^+e^-))$
$[[\mu^-\mu^+]]((\gamma))$
$[[e^+e^-]]$
$[[\mu^-\mu^+]]$
$[[e^+e^-]]((\pi^+\pi^-))$
$[[\pi^+\pi^-]]((\gamma))((\gamma))$
$[[\pi^+\pi^-]]((\pi^\circ))((\gamma))$
$[[\pi^+\pi^-]]$
$[[e^+e^-]]((\pi^\circ))$
$[[\mu^-\mu^+]]((\pi^\circ))$
$[[\mu^-\mu^+]]((\pi^\circ))((\gamma))$

Strange mesons
K^+ $[[\mu^+\nu]]$
$[[\pi^+]]((\pi^\circ))$
$[[\pi^+]]((\pi^+\pi^-))$
$[[\pi^+]]((\pi^\circ))((\pi^\circ))$
$[[\mu^+\nu]]((\pi^\circ))$
$[[e^+\nu]]((\pi^\circ))$
$[[\mu^+\nu]]((\gamma))$
$[[\pi^+]]((\pi^\circ))((\gamma))$
$[[\pi^+]]((\pi^+\pi^-))((\gamma))$
$[[\mu^+\nu]]((\pi^\circ))((\gamma))$
$[[e^+\nu]]((\pi^\circ))((\gamma))$
$[[e^+\nu]]((\pi^\circ))((\pi^\circ))$
$[[e^+\nu]]((\pi^+\pi^-))$
$[[\pi^+]]((e^-\nu\bar{\pi}^+))$
$[[\mu^+\nu]]((\pi^+\pi^-))$
$[[\pi^+]]((\mu^-\nu\bar{\pi}^+))$
$[[e^+\nu]]$
$[[e^+\nu]]((\gamma))$
$[[\mu^+]]((e^+e^-))$
$[[e^+\nu]]((e^+e^-))$
K_s° $[[\pi^+\pi^-]]$
$[[\pi^\circ]]((\pi^\circ))$

$[[\pi^+\pi^-]]((\gamma))$
$[[e^+e^-]]$
$[[\gamma]]((\gamma))$
$[[\pi^\circ]]((\pi^+\pi^-))$
$[[\pi^\circ]]((\pi^\circ))((\pi^\circ))$
K_1° $[[\pi^\circ]]((\pi^\circ))((\pi^\circ))$
$[[\pi^+\pi^-]]((\pi^\circ))$
$[[\pi^+\mu^-\bar{\nu}]]$
$[[\pi^-\mu^+\nu]]$
$[[\pi^+e^-\bar{\nu}]]$
$[[\pi^-e^+\nu]]$
$[[\pi^+\pi^-]]$
$[[\pi^\circ]]((\pi^\circ))$
$[[\pi^+e^-\bar{\nu}]]((\gamma))$
$[[\pi^+\pi^-]]((\gamma))$
$[[\pi^\circ]]((\gamma))((\gamma))$
$[[\gamma]]((\gamma))1$
$[[\mu^-e^+]]((\pi^\circ))$
$[[\mu^-\mu^+]]$
$[[\mu^-\mu^+]]((\gamma))$
$[[\mu^-\mu^+]]((\pi^\circ))$
$[[e^+e^-]]((\gamma))$
$[[\pi^\circ]]((e^+e^-))$
$[[\pi^+\pi^-]]((e^+e^-))$

Charmed strange mesons
F^+ $[[\pi^+]]((\phi))$
$[[\pi^+]]((\eta))((\pi^+\pi^-))$
$[[\pi^+]]((\eta'))((\pi^+\pi^-))$
$[[\rho^+]]((\phi))$

Bottom mesons
B^+ $[[\pi^+]]((D^\circ))$
$[[\pi^+]]((\eta))((\pi^+D^{*-}))$
B° $[[D^\circ]]((\pi^+\pi^-))$
$[[\pi^+D^{*-}]]$

Charmed nonstrange mesons
D^+ $[[\mu^+\nu]]$
$[[\pi^+]]((\pi^+K^-))$
$[[\pi^+]]((\pi^+K^-))((\pi^\circ))$

$[[\pi^+]]((\bar{K}^\circ))$
$[[\pi^+]]((\bar{K}^\circ))((\pi^\circ))$
$[[K^+\nu]]((\bar{K}^\circ))$
$[[e^+\nu]]((\pi^+\pi^-))$
$[[\pi^+\nu]]((K^+K^-))$
$[[K^+]]((\pi^+\pi^-))$
$[[\pi^+]]((\pi^\circ))$
$[[\pi^+]]((\bar{K}^{*\circ}))$
D° $[[K^-\pi^+]]$
$[[K^-\pi^+]]((\pi^\circ))$
$[[\bar{K}^\circ]]((\pi^\circ))$
$[[\pi^+\pi^-]]$
$[[K^+K^-]]$
$[[K^{*-}\pi^+]]$
$[[K^{*\circ}]]((\pi^\circ))$
$[[K^-\rho^+]]$
$[[\bar{K}^\circ]]((\rho^\circ))$
$[[K^-\pi^+]]$
$[[K^-A_2^+]]$

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